Topology and its Applications 27 (1987) 201-212
North-Holland

REFLECTIVE SUBCATEGORIES

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Received 20 May 1986
Revised 25 August 1986

For an abstract category $\mathscr{C}$, several sufficient conditions for the intersection of two or of a larger collection of full reflective subcategories of $\mathscr{C}$ to be reflective are presented. The given criteria are applicable particularly when $\mathscr{C}$ is a concrete topological category.

AMS (MOS) Subj. Class.: Primary 18A40; secondary 18A32, 18B30

reflective subcategory	(strongly) epireflective hull
reflective hull	well-pointed endofunctor
$\mathscr{E}$-localization	orthogonal $\mathscr{E}$-factorization
extremal $\mathscr{D}$-epimorphism	essential monomorphism

1. Introduction

This paper gives a survey of old and new results on (full) reflective subcategories of a category $\mathscr{C}$, especially on the *Intersection Problem* for such subcategories: is $\mathscr{A} \cap \mathscr{B}$ reflective in $\mathscr{C}$ if $\mathscr{A}$ and $\mathscr{B}$ are? In general, the answer is negative. For instance, the reflective subcategories of the even and of the odd numbers in the ordered set of natural numbers, considered as a category, have an empty intersection which is therefore not reflective. Putting this idea into the context of ordinal numbers one obtains a negative answer even when $\mathscr{C}$ is complete and cocomplete (cf. [2]). A more concrete counterexample was discovered by J. Adámek in June 1986 and was communicated to the author after he had completed his work on this article: the topological (!) category of bitopological spaces contains two reflective subcategories with nonreflective intersection. For the category **Top** of topological spaces, the problem is still unsolved (cf. [12]). Also for monadic categories over **Set**, the problem is still open.

Under reasonable conditions on $\mathscr{C}$, the Intersection Problem has a positive solution if one admits only special types of reflective subcategories, and then even intersections of arbitrary collections of subcategories instead of binary ones may be admitted:

(I) *Epireflective subcategories* or, more generally, $\mathscr{E}$-reflective subcategories trivially have reflective intersections as soon as $\mathscr{C}$ allows a characterization of such

0166-8641/87/$3.50

subcategories in terms of closedness under products and certain subobjects or under certain sources (for summaries, see Herrlich, Salicrup and Vázquez [13] and Herrlich [12]).

(II) *Reflective subcategories whose reflector preserves monomorphisms* have intersections of the same type if $\mathscr{C}$ is a well-powered and co-well-powered complete category with amalgamations (cf. Ringel [26]).

(III) The intersection of a *small* family of *α-localizations* of the finitely complete category in which filtered colimits exist and commute with finite limits, is again an α-localization, that is: a reflective subcategory whose reflector preserves finite limits and which is closed under α-filtered colimits, α a regular cardinal (cf. Borceux and Kelly [2]).

At the first glance, the results (II) and (III) do not seem to be interesting for the topologist: a reflective subcategory of **Top** which contains a space with at least two points has a reflector preserving monomorphisms if and only if it is bireflective (i.e., the reflexions are bijections), so we are back to case (I) (cf. Ringel [26]). In addition, F. Cagliari and S. Mantovani recently showed that there are no nontrivial localizations in **Top** (private communication, June 1986). Nevertheless, both papers [26] and [2] provide ideas and techniques which can be adapted to the case of arbitrary reflective subcategories.

We first present an update on Adjoint Functor Theorem methods applied to the Intersection Problem; essentially these were known already in the sixties (cf. [15, 21, 10]). However, a careful re-examination leads us to Theorem 2.2 below which slightly strengthens even the newer result by Harvey [9]. Ringel's results show that a certain condition of co-well-poweredness in Theorem 2.2 can be replaced by the condition that certain bimorphisms have to be essential monomorphisms (Theorem 3.2). Both theorems live on the fact that many properties of a reflective subcategory are inherited by its (strongly) epireflective hull, and vice versa. The forthcoming paper [7] gives a more intrinsic reason for this phenomenon (for $\mathscr{C}=$ **Top**, see also [6]).

Section 4 presents attempts to solve the Intersection Problem by iterative methods, using ordinal powers of well-pointed endofunctors (cf. Kelly [19]). We review a result obtained in [29] (Theorem 4.4) and add another intersection theorem (Theorem 4.6) for reflective subcategories which are closed under colimits of β-chains (i.e., colimits of diagrams over the ordered set $\beta=\{\alpha \mid \alpha<\beta\}$); it follows from a remark in [2].

In Sections 5 and 6 we take advantage of the one-to-one correspondence between reflective subcategories and certain factorization systems, first established by Ringel [25], examined in [24] and [13] under the restriction to epireflective subcategories, and revisited in full generality in [4] and [22]. We recall here in some detail a result by Cassidy, Hébert and Kelly [4] (Theorem 5.2) from which one can derive the existence of small suprema in the 'poset' of all reflective subcategories of a complete and wellpowered category (Theorem 6.3); for localizations, this was proved before by Borceux and Kelly [2].

In this paper, for reasons of brevity and uniformity, we take a conservative approach and impose completeness and smallness conditions on the category $\mathscr{C}$ which contains all our subcategories; '$(\mathscr{E}, \mathscr{M})$-generalizations' may be treated in a later paper. In any case, the topologist can always choose $\mathscr{C}$ to be **Top** or any other topological category over **Set**.

2. Adjoint functor theorem methods

Throughout the paper, every subcategory is full and replete (=isomorpism-closed) and is frequently identified with its class of objects.

The first basic property of a reflective subcategory $\mathscr{D}$ of a category $\mathscr{C}$ is its closedness under the formation of limits in $\mathscr{C}$. It is less elementary to show that the converse proposition is false, particularly for $\mathscr{C} = \mathbf{Top}$ (cf. [18, 11, 16]). A [strongly] epireflective subcategory (that is: the reflexions are [strong] epimorphisms) is also closed under strong monomorphisms [all monomorhpisms]. Here a converse proposition does exist, under suitable conditions on $\mathscr{C}$, the most general and elegant one being that $\mathscr{C}$ comes equipped with a factorization system of sources (cf. [13]). Therefore, it is often advantageous to decompose a given reflective embedding into two epireflective embeddings, a technique first studied extensively by Baron [1].

For a subcategory $\mathscr{D}$ of $\mathscr{C}$, let

$$S(\mathscr{D}) := \{X \in \mathscr{C} \mid \text{there is a monomorphism } X \to D \text{ with } D \in \mathscr{D}\};$$

if $\mathscr{D}$ is reflective with reflexion morphism δ, then

$$S(\mathscr{D}) = \{X \in \mathscr{C} \mid \delta X \text{ is a monomorphism}\}.$$

If, moreover, every morphism in $\mathscr{C}$ factorizes (strong epi, mono), then $S(\mathscr{D})$ is the *strongly epireflective hull* of $\mathscr{D}$ in $\mathscr{C}$, and $\mathscr{D}$ is bireflective in $S(\mathscr{D})$. Completely analogous assertions hold for

$$\bar{S}(\mathscr{D}) := \{X \in \mathscr{C} \mid \text{there is a strong monomorphism } X \to D \text{ with } D \in \mathscr{D}\}$$

which is the epireflective hull of $\mathscr{D}$ if $\mathscr{D}$ is reflective and if $\mathscr{C}$ has (epi, strong mono)-factorizations. Trivially,

$$\mathscr{D} \subseteq \bar{S}(\mathscr{D}) \subseteq S(\mathscr{D}) \subseteq \mathscr{C}.$$

2.1. Proposition (Herrlich [11], Hoffmann [14]). *For a subcategory $\mathscr{D}$ of a complete, well-powered and co-well-powered category $\mathscr{C}$, the following assertions are equivalent:*

(i) *$\mathscr{D}$ is reflective and co-well-powered,*
(ii) *$\mathscr{D}$ is closed under limits and $S(\mathscr{D})$ is co-well-powered,*
(iii) *$\mathscr{D}$ is closed under limits and $\bar{S}(\mathscr{D})$ is co-well-powered.*

Proof. For (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) see [14]. For (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) one notices first that epimorphisms in $\mathscr{D}$ are epimorphisms in $S(\mathscr{D})$ and $\bar{S}(\mathscr{D})$, hence co-well-poweredness of $\mathscr{D}$ follows trivially. Since $S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$ is closed under products and [strong] monomorphisms in $\mathscr{C}$ it is strongly epireflective [epireflective] in $\mathscr{C}$. Therefore, it suffices to show that $\mathscr{D}$ is reflective in $S(\mathscr{D})$ and in $\bar{S}(\mathscr{D})$.

One first notices that $S(\mathscr{D})$ and $\bar{S}(\mathscr{D})$ are complete and well-powered, and that $\mathscr{D}$ is closed under products and regular monomorphisms in $S(\mathscr{D})$ and in $\bar{S}(\mathscr{D})$. Next, by standard techniques one has that every $f: X \to D$ with $X \in S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$ and $D \in \mathscr{D}$ factorizes as $f = me$ with $m: D' \to D$ a monomorphism in $\mathscr{D}$ and e an epimorphism in $S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$ (which, moreover, does not admit any nontrivial factorization $e = ng$ with a monomorphism n in $\mathscr{D}$). Therefore, a representative small set of epimorphisms in $S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$ with codomain in $\mathscr{D}$ will form a solution set for the object X (cf. [10]). $\square$

The morphism $e: X \to D'$ constructed in the proof above is an *extremal $\mathscr{D}$-epimorphism*, that is: $ce = de$ implies $c = d$ for any two morphisms c, d in $\mathscr{D}$, and e factorizes only trivially through a monomorphism of $\mathscr{D}$. Therefore, Proposition 2.1 actually gives us the following stronger version of a result due to Harvey [9]:

2.2. Theorem. *In a complete, well-powered and co-well-powered category $\mathscr{C}$, a limit-closed subcategory $\mathscr{D}$ is reflective if every object $X \in \bar{S}(\mathscr{D})$ admits only a small set of nonisomorphic extremal $\mathscr{D}$-epimorphisms with domain X; the latter condition is satisfied if $S(\mathscr{D})$ or $\bar{S}(\mathscr{D})$ is co-well-powered.*

The theorem applies in particular to the case that $\mathscr{D}$ is the intersection of any collection of reflective subcategories of $\mathscr{C}$ (which is trivially closed under limits).

Remark. Under the smallness condition of Theorem 2.2 one still gets, as a necessary condition for reflectivity, that $\mathscr{D}$ is co-well-powered with respect to strong (or, equivalently, extremal) epimorphisms.

3. Ringel's result revisited

A monomorphism m in $\mathscr{C}$ is *essential* if f is a monomorphism whenever fm is one. In Proposition 3.1 and Theorem 3.2, we shall assume that (certain) bimorphisms (both: epi- and monomorphisms) are essential. This assumption is satisfied when $\mathscr{C}$ has (weak) amalgamations, that is: for every pair of monomorphisms m, n with common domain there are monomorphisms u, v with $um = vn$. Indeed, if m is a bimorphism, if $n = fm$ is a monomorphism, and if u, v are chosen as above, then $u = vf$ since m is an epimorphism; so f must be a monomorphism since u is one.

3.1. Proposition (Ringel [26]). *Let $\mathscr{C}$ be complete, well-powered and co-well-powered, and let the subcategory $\mathscr{D}$ be closed under limits. Then $\mathscr{D}$ is reflective in $\mathscr{C}$ if bimorphisms in $S(\mathscr{D})$ or $\bar{S}(\mathscr{D})$ are essential in $S(\mathscr{D})$ or $\bar{S}(\mathscr{D})$ respectively.*

Proof. It suffices to show that $\mathscr{D}$ is reflective in $S(\mathscr{D})$ or $\bar{S}(\mathscr{D})$. So let X be in $S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$, pick a [strong] monomorphism $f: X \to D$ with $D \in \mathscr{D}$, and factorize f as in the proof of Proposition 2.1 through an extremal $\mathscr{D}$-epimorhphism $e: X \to D'$ and a monomorphism $m: D' \to D$. To see that e is a $\mathscr{D}$-reflexion of X, one considers a morphism $g: X \to D''$, $D'' \in \mathscr{D}$, and factorizes the induced morphism $h: X \to D' \times D''$ with $p_1 h = e$, $p_2 h = g$ in the form $h = m'e'$ with an extremal $\mathscr{D}$-epimorphism e' and a monomorphism m' in $\mathscr{D}$. Since $mp_1m'e' = me = f$ is a monomorphism, also e and e' are monomorphisms. So e' is a bimorphism in $S(\mathscr{D})$ $[\bar{S}(\mathscr{D}]$ and therefore essential. Hence, $p_1 m'$ is a monomorphism since $p_1 m'e' = e$ is one. Extremality of e then forces $p_1 m'$ to be an isomorphism. So we obtain the factorization $p_2 m' \, (p_1 m')^{-1} e = g$. $\square$

Remarks. (1) If $S(\mathscr{D})$ or $\bar{S}(\mathscr{D})$ has the property that bimorphisms are essential also $\mathscr{D}$ has that property.

(2) The example $\mathscr{C} = \mathbf{Top}$, $\mathscr{D} = \mathbf{CompHaus}$ (and $\bar{S}(\mathscr{D}) = \mathbf{ComplRegHaus}$) shows that essentiality of bimorphisms is not a necessary condition for reflectivity of $\mathscr{D}$: consider the reflexions $X \to \beta X$. (For recent investigations on essential reflexions see [8].) However: if the reflexions are essential in $S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$ and if $\mathscr{D}$ has the property that bimorphisms are essential, also $S(\mathscr{D})$ $[\bar{S}(\mathscr{D})]$ has that property.

In the proof of Proposition 3.1, if we work with $\bar{S}(\mathscr{D})$, e and e' are in fact strong monomorphisms in $\mathscr{C}$ and extremal $\mathscr{D}$-epimorphisms in $\bar{S}(\mathscr{D})$. So we only need that such morphisms are essential in $\bar{S}(\mathscr{D})$:

3.2. Theorem. *In a complete, well-powered and co-well-powered category $\mathscr{C}$, a limit-closed subcategory $\mathscr{D}$ is reflective in $\mathscr{C}$ if extremal $\mathscr{D}$-epimorphisms in $\bar{S}(\mathscr{D})$ which, at the same time, are strong monomorphisms in $\mathscr{C}$, are essential in $\bar{S}(\mathscr{D})$.*

Again, this theorem can be applied when $\mathscr{D}$ is the intersection of a collection of reflective subcategories $\mathscr{A}_i$, $i \in I$. However, the weakness of Theorems 2.2 and 3.2 then lies in the fact that only closedness under limits for each $\mathscr{A}_i$ is used. In the following, reflectivity of $\mathscr{A}_i$ will be used to a greater extent.

4. Iterative methods

A reflective subcategory $\mathscr{A}$ of a cateogry $\mathscr{C}$ gives rise to a *well-pointed endofunctor* (r, ρ) of $\mathscr{C}$, that is: a functor $r: \mathscr{C} \to \mathscr{C}$ and a natural transformation $\rho: 1_{\mathscr{C}} \to r$ with $r\rho = \rho r$; one has

$$\mathscr{A} = \mathrm{Fix}(r, \rho) := \{X \in \mathscr{C} \mid \rho X \text{ isomorphism}\}.$$

We note that, for every well-pointed endofunctor (r, ρ) of $\mathscr{C}$, ρX is already an isomorphism if it is a split-monomorphism (cf. [2, 29]).

Vice versa, $\mathrm{Fix}(r, \rho)$ is, for every well-pointed endofunctor (r, ρ) of $\mathscr{C}$, limit-closed and often reflective in $\mathscr{C}$. To see this one forms the ordinal chain

$$\rho_\alpha^\beta : r^\alpha \to r^\beta, \quad \alpha \leq \beta,$$

with $r^0 = 1_{\mathscr{C}}$, $r^{\beta+1} = rr^\beta$, $\rho_\alpha^{\beta+1} = \rho r^\beta \cdot \rho_\alpha^\beta$, and

$$r^\beta = \mathrm{colim}_{\alpha<\beta}\, r^\alpha \quad \text{with canonical injections } \rho_\alpha^\beta,$$

in case β is a limit ordinal; of course, it must be assumed here that $\mathscr{C}$ has the respective colimits. Putting $\rho^\beta = \rho_0^\beta$ one obtains, for each ordinal β, a well-pointed endofunctor (r^β, ρ^β) of $\mathscr{C}$ with

$$\mathrm{Fix}(r^\beta, \rho^\beta) = \mathrm{Fix}(r, \rho)$$

(cf. [2, 29]). As in [29], we call (r, ρ) *co-well-powered* if, for every $X \in \mathscr{C}$, there are ordinals $\alpha < \beta$ such that $\rho_\alpha^\beta X$ is an isomorphism; this implies that $\rho_\alpha^{\alpha+1} X$ is an isomorphism, so $r^\alpha X \in \mathrm{Fix}(r, \rho)$. Then $\rho^\alpha X : X \to r^\alpha X$ is even a reflexion of X into $\mathrm{Fix}(r, \rho)$. So one has the following:

4.1. Proposition. *In a category $\mathscr{C}$ with colimits of chains, a subcategory $\mathscr{A}$ is reflective if and only if $\mathscr{A} = \mathrm{Fix}(r, \rho)$ for some co-well-powered and well-pointed endofunctor (r, ρ) of $\mathscr{C}$.*

Remarks. (1) The above proposition has been formulated in [29] in the context of *prereflections* but all arguments remain valid if one works with (the slightly more general concept of) well-pointed endofunctors instead (see also Remark (2) after Proposition 5.1). Borceux and Kelly consider the case that ρ has rank α; then $\rho^\alpha X$ is a reflexion for every X, so α does not depend on X (see [2, Theorem 4.8]; see also Theorem 4.6).

(2) Proposition 4.1 does not exclude the possibility that a reflective subcategory is presented as $\mathrm{Fix}(r, \rho)$ with a well-pointed endofunctor (r, ρ) which is not co-well-powered (see [29]).

Despite of the previous remark, very often it is advantageous to work with the induced well-pointed endofunctor rather than to consider the reflective subcategory directly as the two basic Lemmata 4.2 and 4.5 will indicate.

4.2. Lemma (Wolff [30]). *For the well-pointed endfunctor (s, σ) of $\mathscr{C}$ and the adjunction $F \dashv G : \mathscr{A} \to \mathscr{C}$ with co-unit ε, let (t, τ) be defined pointwise by the push-out*

$$\begin{tikzcd} FG \arrow[r, "F\sigma G"] \arrow[d, "\varepsilon"'] & FsG \arrow[d] \\ 1_{\mathscr{A}} \arrow[r, "\tau"] & t. \end{tikzcd} \tag{1}$$

Then (t, τ) is a well-pointed endofunctor of $\mathscr{A}$ with $\mathrm{Fix}(t, \tau) = G^{-1}\mathrm{Fix}(s, \sigma)$.

In case $\mathscr{A}$ is a reflective subcategory of $\mathscr{C}$ the co-unit of the induced adjunction is an isomorphism, so the push-out (1) exists trivially. Moreover, if σ is a pointwise [strong] epimorphism, also τ is one, so (t, τ) is co-well-powered if the category $\mathscr{A}$ is co-well-powered [with respect to strong epimorphisms]. So one has (cf. [29, Corollary 6]) the following proposition:

4.3. Proposition. *For a category $\mathscr{C}$ with colimits of chains, for $\mathscr{A}$ reflective and co-well-powered [with respect to strong epimorphisms], and for $\mathscr{B}$ [strongly] epireflective in $\mathscr{C}$, also $\mathscr{A} \cap \mathscr{B}$ is [strongly] epireflective in $\mathscr{A}$ and, a fortiori, reflective in $\mathscr{C}$.*

If $\mathscr{B}$ is reflective but not epireflective we can decompose the $\mathscr{B} \to \mathscr{C}$ into two epireflections, using $S(\mathscr{B})$ or $\bar{S}(\mathscr{B})$ of Section 2, and apply Proposition 4.3 twice to obtain the following theorem.

4.4. Theorem (Tholen [26]). *In a category $\mathscr{C}$ with colimits of chains the intersection of two reflective subcatergories $\mathscr{A}$ and $\mathscr{B}$ is reflective if condition* (a) *or* (b) *holds:*

(a) *$\mathscr{C}$ has (epi, strong mono)-factorizations, and $\mathscr{A}$ and $\mathscr{A} \cap \bar{S}(\mathscr{B})$ are co-well-powered.*

(b) *$\mathscr{C}$ has (strong epi, mono)-factorizations, $\mathscr{A}$ is co-well-powered with respect to strong epimorphisms, and $\mathscr{A} \cap S(\mathscr{B})$ is co-well-powered.*

Iterative methods are particularly useful when one deals with subcategories with rank, that is: subcategories which are closed under α-filtered colimits (for a regular cardinal α); in fact, closedness under colimits of α-chains suffices as we shall see in Theorem 4.6. For that, we need an easy but important lemma:

4.5. Lemma (Borceux and Kelly [2]). *For well-pointed endofunctors (r, ρ) and (s, σ) of $\mathscr{C}$, also (sr, δ) with*

$$\delta = (1 \xrightarrow{\rho} r \xrightarrow{\sigma r} sr) = (1 \xrightarrow{\sigma} s \xrightarrow{s\rho} sr)$$

is well-pointed, and $\mathrm{Fix}(sr, \delta) = \mathrm{Fix}(r, \rho) \cap \mathrm{Fix}(s, \sigma)$.

Remark (Pedicchio). If ρX and σX are reflexions of X into $\mathscr{A} = \mathrm{Fix}(r, \rho)$ and $\mathscr{B} = \mathrm{Fix}(s, \sigma)$ respectively, δX need not be a reflexion of X into $\mathscr{A} \cap \mathscr{B}$: consider $\mathscr{C} = \mathbf{Top}$, $\mathscr{A} = T_1 - \mathbf{Top}$, and $\mathscr{B} = \mathbf{Sober}$ (cf. [17, p. 44]).

If r and s preserve colimits of β-chains, also sr does so, and (sr, δ) is therefore co-well-powered. From Proposition 4.1 and Lemma 4.5 one then obtains the following theorem which is contained in a remark in [2, before Theorem 5.3].

4.6. Theorem. *Let $\mathscr{C}$ admit colimits of β-chains (β any infinite ordinal), and let the reflective subcategories $\mathscr{A}$ and $\mathscr{B}$ be closed under these colimits. Then also $\mathscr{A} \cap \mathscr{B}$ is reflective in $\mathscr{C}$ (and trivially closed under colimits of β-chains).*

5. Relationship with factorization systems

In order to study the 'poset' REFL($\mathscr{C}$) of all reflective subcategories of a category $\mathscr{C}$ it is useful to study the relationship of reflective subcategories and factorization systems first. For an object A and a morphism $p: U \to V$ in $\mathscr{C}$ one writes (cf. [5])

$$p \perp A$$

if every $g: U \to A$ admits a factorization $tp = g$ with a unique t. By

$$\mathscr{A}^{\perp} := \{p \mid p \perp A \text{ for all } A \in \mathscr{A}\},$$

$$\mathscr{E}_{\perp_1} := \{A \mid p \perp A \text{ for all } p \in \mathscr{E}\}$$

one obtains a Galois correspondence between subclasses $\mathscr{A}$ of objects and subclasses $\mathscr{E}$ of morphisms of $\mathscr{C}$. (These are always assumed to be closed under composition with isomorphisms.) Reflective $\mathscr{A}$'s are closed under this correspondence, and they correspond one-to-one to those closed $\mathscr{E}$'s for which every object X in $\mathscr{C}$ has an *$\mathscr{E}$-localization* (in the sense of [3, 25]), that is: a morphism $e: X \to A$ in $\mathscr{E}$ such that for every $p: U \to V$ in $\mathscr{E}$ and every $g: U \to X$ there is a unique t with $tp = eg$. 'Closedness' of $\mathscr{E}$ can be fully characterized in this context:

5.1. Proposition (cf. [22]). *For every category $\mathscr{C}$,* REFL($\mathscr{C}$) *is anti-isomorphic to the partially ordered collection of all classes $\mathscr{E} \subseteq$* Mor $\mathscr{C}$ *which satisfy the following conditions*:

(A) *$qp \in \mathscr{E}$ and $q \in \mathscr{E}$ implies $p \in \mathscr{E}$,*
(B) *$p \in \mathscr{E}$ and $q \in \mathscr{E}$ implies $qp \in \mathscr{E}$,*
(C) *every object in $\mathscr{C}$ has an $\mathscr{E}$-localization.*

Remarks. (1) Proposition 5.1 is a slight generalization of a result by Cassidy, Hébert and Kelly [4] which in turn, generalizes and strengthens earlier work by Ringel [25]. For a reflective $\mathscr{A}$ with reflector r, these authors use the presentation

$$\mathscr{A}^{\perp} = \{p \mid r(p) \text{ is an isomorphism}\}$$

from which properties (A), (B), (C) follow trivially.

(2) Korostenski and Tholen [22] have characterized those subclasses $\mathscr{A}$ which correspond to the subclasses $\mathscr{E}$ which satisfy (C) and

(A′) $\quad qp \in \mathscr{E}$ and $sq \in \mathscr{E}$ implies $p \in \mathscr{E}$

((A′) is equivalent to (A) in the presence of (B) and (C)). Those are subclasses $\mathscr{A}$ with $\mathscr{A} = \mathrm{Fix}(r, \rho)$ for a so-called *regular prereflection* (see Remark (1) after Proposition 4.1).

(3) Recall that a pair (e, m) is an *orthogonal $\mathscr{E}$-factorization* of $f = me$ if e belongs to $\mathscr{E}$ and m to

$$\mathscr{E}_{\perp_2} := \{s \mid p \perp s \text{ for all } p \in \mathscr{E}\};$$

here $p \perp s$ means that $hp = sg$ implies the unique existence of a morphism t with $tp = g$ and $st = h$. (The use of the same symbol $\perp$ again is justified by the fact that in both cases we have special instances of a more general relation between morphisms and factorizations of sources which, by the way, also covers the defining property of an $\mathscr{E}$-localization; see [28] and [22].)

The $\mathscr{E}$-part of an orthogonal $\mathscr{E}$-factorization of $X \to 1$ (where 1 is a terminal object of $\mathscr{C}$) is an $\mathscr{E}$-localization of X. Vice versa, in the presence of (B), an $\mathscr{E}$-localization of X gives an orthogonal $\mathscr{E}$-factorization of $X \to 1$. The following theorem deals with the question when an arbitrary morphism $f: X \to Y$ allows such a factorization and will be used in Theorem 6.3.

5.2. Theorem (Cassidy, Hébert and Kelly [4]). *Let $\mathscr{C}$ be complete and well-powered with respect to strong monomorphisms. Then* REFL($\mathscr{C}$) *is anti-isomorphic to the partially ordered collection of all subclasses $\mathscr{E} \subseteq \mathrm{Mor}\, \mathscr{C}$ which satisfy* (A), (B) *of Proposition* 5.1 *and*

(C′) *every morphism in $\mathscr{C}$ has an orthogonal $\mathscr{E}$-factorization.*

In fact, it suffices to assume that $\mathscr{C}$ is finitely complete and admits arbitrary intersections of strong monomorphisms.

Proof. For a reflective $\mathscr{A}$ with reflexion $\rho: 1 \to r$ and $\mathscr{E} := \mathscr{A}^{\perp} = \{p \mid r(p) \text{ isomorphism}\}$ one has to show (C′). So let $f: X \to Y$ be a $\mathscr{C}$-morphism, and let $m: Z \to Y$ be the pullback of $r(f)$ along ρY:

$$\begin{tikzcd}
X \arrow[dr, dashed, "w"] \arrow[drr, "\rho X"] \arrow[ddr, "f"'] & & \\
& Z \arrow[r] \arrow[d, "m"] & rX \arrow[d, "r(f)"] \\
& Y \arrow[r, "\rho Y"'] & rY
\end{tikzcd} \tag{3}$$

It is easy to see that $r(f) \in \mathscr{M} := \mathscr{E}_{\perp_2}$, so also $m \in \mathscr{M}$. The induced morphism w factorizes through a least strong monomorphism n in $\mathscr{M}$ in the form $w = ne$. In order to see that (e, mn) is an orthogonal $\mathscr{E}$-factorization of f one just needs to verify that $r(e)$ is an isomorphism, so $e \in \mathscr{E}$; this is left to the reader. □

6. Lattice properties of REFL($\mathscr{C}$)

In this section we first want to justify that, lattice-theoretically, the Intersection Problem is, after all, the 'right problem'. For that we need:

6.1. Lemma. *Let $\mathscr{C}$ be complete and well-powered. Then the limit-closure $L(D)$ of the subcategory $\{D\}$ for a single object D in $\mathscr{C}$ is also the reflective hull of $\{D\}$ in $\mathscr{C}$, and D is a strong cogenerator of $L(D)$.*

Proof. Let $\Pi(D)$ denote the subcategory of all powers of D. Since $\bar{S}(\Pi(D))$ is closed under limits in $\mathscr{C}$ one has $L(D) \subseteq \bar{S}(\Pi(D))$. By definition, $\{D\}$ is a strong cogenerator of $\bar{S}(\Pi(D))$ and, a fortiori, of $L(D)$. So everything follows from the Special Adjoint Functor Theorem. □

Remarks. (1) The above lemma appears as a remark in [26]. It holds more generally for any small subcategory $\mathscr{D}$ of $\mathscr{C}$ (as D may be replaced by $\mathscr{D}$ in the proof; see also [28]). In addition one can state that, due to the existence of a strong cogenerator, $L(\mathscr{D})$ is also co-well-powered provided $\mathscr{C}$ is finitely cocomplete (cf. [23, pp. 111–112; 2, Prop. 6.1]).

(2) By Proposition 2.1, an arbitrary subcategory $\mathscr{D}$ has a reflective hull in $\mathscr{C}$ (which is equal to the limit-closure $L(\mathscr{D})$) if $\mathscr{C}$ satisfies the conditions of 2.1 and if $S(L(\mathscr{D}))$ or $\bar{S}(L(\mathscr{D}))$ is co-well-powered (cf. [11] and, for a different criterion [9]).

(3) In Lemma 6.1 it suffices to assume well-poweredness just with respect to strong monomorphisms, or just the existence of arbitrary intersections of strong monomorphisms. A full proof of the lemma in this slightly more general form will appear in [20].

6.2. Proposition (Kelly [20]). *Let $\mathscr{C}$ be as in Lemma 6.1, and let $(\mathscr{A}_i)_{i\in I}$ be any collection of reflective subcategories of $\mathscr{C}$ which has an infimum $\mathscr{A}$ in* REFL($\mathscr{C}$). *Then $\mathscr{A} = \bigcap_{i\in I} \mathscr{A}_i$.*

Proof. Trivially one has '$\subseteq$'. Vice versa, let D be in every $\mathscr{A}_i$. For the reflective hull $L(D)$ one then has $L(D) \subseteq \mathscr{A}_i$ for every $i \in I$, hence $L(D) \subseteq \mathscr{A}$. Therefore $D \in \mathscr{A}$ follows. □

In order to solve the Intersection Problem, by Proposition 6.2 we 'just' had to show the existence of arbitrary infima in REFL($\mathscr{C}$). For that it would suffice to show the existence of arbitrary suprema. These, in turn, exist according to Proposition 5.1 (or Theorem 5.2) whenever one can prove that the intersection $\mathscr{E}$ of any collection $(\mathscr{E}_i)_{i\in I}$ of subclasses of Mor $\mathscr{C}$ satisfies conditions (A), (B) and (C) (or (C′)) of Section 5 if each $\mathscr{E}_i$ does. By a careful analysis of a corresponding result by Borceux and Kelly (cf. [2, Theorem 3.1]) in the context of localizations one succeeds to do so provided I is small. This leads to the following theorem, a full proof of which will appear in [20].

6.3. Theorem (Kelly [20]). *If $\mathscr{C}$ is complete and well-powered with respect to strong monomorphisms, then any small collection of reflective subcategories of $\mathscr{C}$ has a supremum in* REFL($\mathscr{C}$).

Proof (sketch). Properties (A) and (B) of Section 5 are trivially stable under intersection. For (C′), consider a morphism $f: X \to Y$ in $\mathscr{C}$ and form, for each $i \in I$, its orthogonal $\mathscr{E}_i$-factorization (e_i, m_i). As I is small we have a multiple pullback

$m: Y' \to Y$ of $(m_i)_{i\in I}$ with projections p_i, $i \in I$. The induced morphism $f': X \to Y'$ *with* $p_i f' = e_i$, $i \in I$, has again, for each $i \in I$, an orthogonal $\mathscr{E}_i$-factorization (e_i', m_i'). With the proof of Theorem 5.2 one can show that each m_i' is a strong monomorphism. An ordinal iteration of the above factorization procedure will therefore 'stop' eventually and then lead to an orthogonal $\mathscr{E}$-factorization of f. $\square$

Remarks. (1) From Theorem 6.3 one derives a positive solution of the Intersection Problem in case REFL($\mathscr{C}$) is small. This is trivially true for $\mathscr{C}$ = **Set**; but there are less trivial examples of categories for which REFL($\mathscr{C}$) is not only small but just consists of its top and bottom element: $\mathscr{C} = \mathbf{Ban}^{\mathrm{op}}$ (cf. [27]).

(2) (Ringel [26]). In general, suprema in REFL($\mathscr{C}$) cannot be found by just forming the limit-closure of the union of the reflective subcategories: every limit-closed but nonreflective subcategory $\mathscr{D}$ of $\mathscr{C}$ is trivially the limit-closure of $\bigcup_{D\in\mathscr{D}} L(D)$ where each $L(D)$ is reflective according to Lemma 6.1. As mentioned already in Section 2, even $\mathscr{C}$ = **Top** admits such a subcategory $\mathscr{D}$.

Note added in proof. Subsequently, the Intersection Problem for **Top** has been solved to the negative by V. Trnková, J. Adámek and J. Rosický (communication given in January 1987 at the Winter School in Srni, Czechoslovakia).

Acknowledgement

I should like to thank the referee for various helpful directions which led to improvements of the paper.

References

[1] S. Baron, Reflectors as compositions of epi-reflectors, Trans. Amer. Math. Soc. 136 (1969) 499-508.
[2] F. Borceux and G.M. Kelly, On locales of localizations, J. Pure Appl. Algebra 46 (1987) 1-34.
[3] A.K. Bousfield, Constructions of factorization systems in categories, J. Pure Appl. Algebra 9 (1977) 207-220.
[4] C. Cassidy, M. Hébert, and G.M. Kelly, Reflective subcategories, localizations and factorization systems, J. Austral. Math. Soc. Ser. A38 (1985) 287-329.
[5] P.J. Freyd and G.M. Kelly, Categories of continuous functors, I, J. Pure Appl. Algebra 2 (1972) 169-191; Erratum ibid. 4 (1974) 121.
[6] E. Giuli and M. Hušek, A diagonal theorem for epireflective subcategories of Top and cowell-poweredness, Annali di Matematica Pura et Appl. 145 (1986) 337-346.
[7] E. Giuli, S. Mantovani and W. Tholen, Objects with closed diagonals, J. Pure Appl. Algebra, to appear.
[8] A.W. Hager and J.J. Madden, Essential reflections versus minimal embeddings, J. Pure Appl. Algebra 37 (1985) 27-32.
[9] J.M. Harvey, Reflective Subcategories, Illinois J. Math. 29 (1985) 365-369.
[10] H. Herrlich, Topologische Reflexionen und Coreflexionen, Lecture Notes Math. 78 (Springer, Berlin/Heidelberg/New York, 1968).
[11] H. Herrlich, Epireflective subcategories of Top need not be cowellpowered, Comm. Math. Univ. Carolinae 16 (1975) 713-715.

[12] H. Herrlich, Categorical Topology 1971-1981, in: J. Novak, ed., Proc. Fifth Prague Topol. Symp. 1981 (Heldermann Verlag, Berlin, 1982) 279-383.

[13] H. Herrlich, G. Salicrup and R. Vázquez, Dispersed factorization structures, Can. J. Math. 31 (1979) 1059-1071.

[14] R.E. Hoffmann, Co-well-powered reflective subcategories, Proc. Amer. Math. Soc. 90 (1984) 45-46.

[15] J.R. Isbell, Structure of categories, Bull. Amer. Math. Soc. 72 (1966) 619-655.

[16] J.R. Isbell, A closed non-reflective subcategory of compact spaces, unpublished manuscript.

[17] P.T. Johnstone, Stone Spaces (Cambridge University Press, Cambridge, 1982).

[18] V. Kannan and M. Rajagopalan, Constructions and applications of rigid spaces I, Advances in Math. 29 (1978) 89-130.

[19] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, Bull. Austral. Math. Soc. 22 (1980) 1-83.

[20] G.M. Kelly, On the ordered set of reflective subcategories, in preparation.

[21] J.F. Kennison, A note on reflection maps, Illinois J. Math. 11 (1967) 404-409.

[22] M. Korostenski and W. Tholen, On left-cancellable classes of morphisms, Comm. Algebra 14 (1986) 741-766.

[23] B. Pareigis, Categories and Functors (Academic Press, New York/London, 1970).

[24] D. Pumplün, Universelle und spezielle Probleme, Math. Ann. 198 (1972) 131-146.

[25] C.M. Ringel, Diagonalisierungspaare I, Math. Z. 117 (1970) 248-266.

[26] C.M. Ringel, Monofunctors as reflectors, Trans. Amer. Math. Soc. 161 (1971) 293-306.

[27] Z. Samadeni and T. Swirszcz, Reflective and coreflective subcategories of categories of Banach spaces and Abelian groups, Bull. Acad. Polonaise des Sciences 25 (1977) 1105-1107.

[28] W. Tholen, Factorizations, localizations, and the Orthogonal Subcategory Problem, Math. Nachr. 114 (1983) 63-85.

[29] W. Tholen, Prereflections and reflections, Comm. Algebra 14 (1986) 717-740.

[30] H. Wolff, Free monads and the Orthogonal Subcategory Problem, J. Pure Appl. Algebra 13 (1978) 233-242.